Block Cipher Cryptanalysis - Part 1

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Overview

Lesson contents

- Basics of Block Cipher Cryptanalysis
- Algebraic Known Plaintext Attack of Block Ciphers w/o Sboxes
  - The *Trivial* cipher
  - Algebraic Cryptanalysis of *Trivial*
- Statistical Known Plaintext Attack: Linear Cryptanalysis
  - The *Simple* cipher
  - Linear Cryptanalysis of *Simple*

Block Cipher Cryptanalysis

Approaches

- Modern block cipher cryptanalysis aims at either recovering the key with an effort smaller than bruteforce
- *Known-ciphertext-only* cryptanalysis is a symptom of a deeply flawed cipher (Caesar, Vigenére, Playfair *et al.*)
- We will tackle techniques requiring either a *known plaintext* or a *related (chosen) plaintext* assumption
- Block cipher cryptanalyses are usually split into two families: algebraic and statistical

Algebraic Analysis

A cipher can be seen as a *large* set of Boolean simultaneous equations involving key bits (variables), ptx and ctx bits (known terms).

Either one or both of the following findings should be obtained efficiently to have a successful cryptanalysis

- A solution (in closed form), for the simultaneous equations yields the key from one (or more) ptx/ctx pairs
- The discovery that only a subset of the key bits are actually involved in the simultaneous equations, while a portion of them self-cancel
  - Only the involved key bits are effectively used by the cipher in encryption

Statistical Analysis

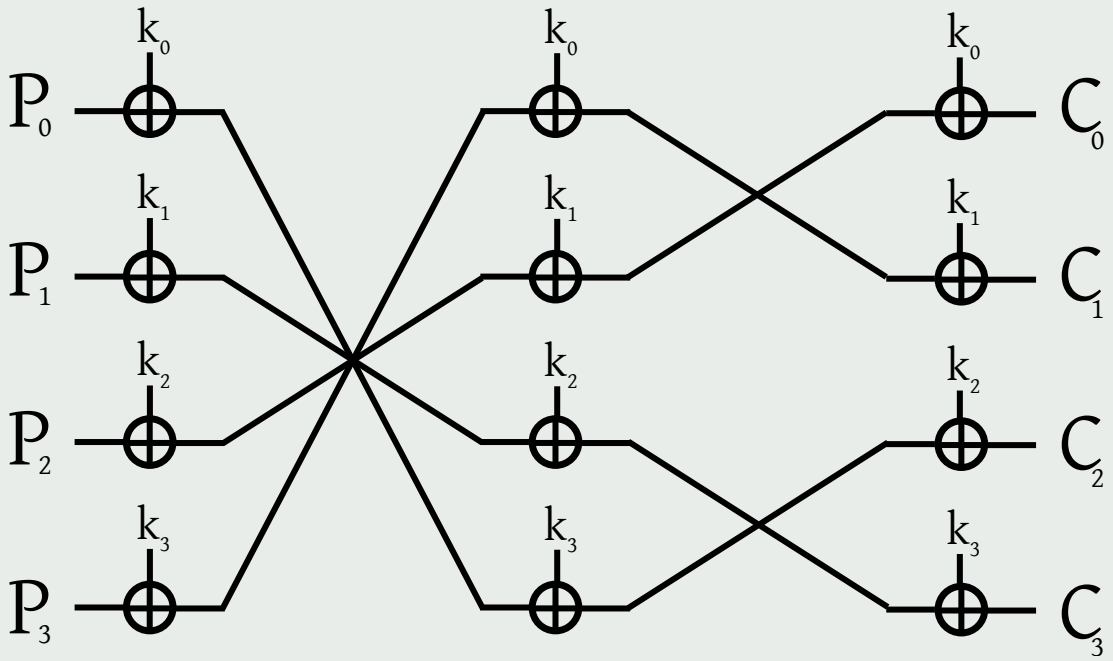
- A perfect cipher is be able to produce a ctx independent from the ptx
- All the computationally secure ones produce a ctx where the probability that a bit is either one or zero is biased ($= \frac{1}{2} \pm \epsilon$) by the ptx value
- Thus, a relation r between ptx and ctx bits may hold with bias $\epsilon \neq 0$
  - E.g., "even numbers of 1s in the ptx $\rightarrow$ even number of 1s in the ctx"
- Goal: find such relations between ptx and ctx, and exploit the statistical biases to extract a portion of the key

Block Cipher Cryptanalysis

Starting out

- From now on, *linear* will be referring to linearity w.r.t. the addition modulo 2, i.e., eXclusive-OR (`xor`)
- Recall that the multiplication mod 2 is the Boolean "`and`" operation and that $x^2 \bmod 2 = x$ if you only care for solutions in $\{0, 1\}$
- First step: observe that any linear cipher is trivial to cryptanalyze
  - Obtaining the key with a *known plaintext attack* = solving a set of simultaneous linear equations
- First example: the *Trivial* cipher, only has *diffusion* and *key addition*

Trivial Cipher



- 4-bit block, 4-bit key
- Key schedule: a repetition of the key

Trivial Cryptanalysis: Algebraic Approach

- Rewrite the relations between ptx, ctx and key bits as equations with binary coefficients:

$$\begin{aligned} C_0 &= P_2 \oplus k_2 \oplus k_1 \oplus k_0 \\ C_1 &= P_3 \oplus k_3 \oplus k_0 \oplus k_1 \\ C_2 &= P_0 \oplus k_0 \oplus k_3 \oplus k_2 \\ C_3 &= P_1 \oplus k_1 \oplus k_2 \oplus k_3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} P_2 \oplus C_0 &= k_2 \oplus k_1 \oplus k_0 \\ P_3 \oplus C_1 &= k_3 \oplus k_0 \oplus k_1 \\ P_0 \oplus C_2 &= k_0 \oplus k_3 \oplus k_2 \\ P_1 \oplus C_3 &= k_1 \oplus k_2 \oplus k_3 \end{aligned}$$

- This is a linear simultaneous eq. set, where the ptxs and ctxs are known: just solve it for k_0, k_1, k_2, k_3
- Any cipher made only with a linear *key addition* and a linear *diffusion* layer can be broken with the same technique

Trivial Cryptanalysis: Statistical Approach

- Since in *Trivial* $C_0 = P_2 \oplus k_2 \oplus k_1 \oplus k_0$:
  - If the relation $\mathbf{r} : \mathbf{C_0} = \mathbf{P_2}$ holds, then either:
    1. $k_0 = k_2 = k_1 = 0$
    2. $k_0 = k_1 = 1, k_2 = 0$
    3. $k_0 = k_2 = 1, k_1 = 0$
    4. $k_1 = k_2 = 1, k_0 = 0$
  - If the relation $\mathbf{r'} : \mathbf{C_0} \neq \mathbf{P_2}$ holds, then either:
    1. $k_0 = k_2 = k_1 = 1$
    2. $k_0 = k_1 = 0, k_2 = 1$
    3. $k_0 = k_2 = 0, k_1 = 1$
    4. $k_1 = k_2 = 0, k_0 = 1$
- In both cases, looking at the value of a ptx-ctx bit pair, out of $2^3 = 8$ possible values for k_0, k_1, k_2 only 4 possibilities are left
- This is equivalent to a keyspace reduction by a factor of 2, and can always be done since either r or r' will hold with $Pr(r) = 1$.

Nonlinear Relations

Enter the S-Boxes

- Sound ciphers have nonlinear components (S-Boxes) preventing efficient closed form solution of simultaneous equations
  - Solving simultaneous equations with deg. > 1 in $\mathbb{Z}_p$ is NP-Complete
- Without S-Boxes, the linear relations between ptx and ctx bits always hold (i.e., with probability $=1$)
- With S-Boxes, linear relations may still hold with a probability $\neq \frac{1}{2}$

Key Idea of Linear Cryptanalysis

1. Exploit the relations holding with a bias (i.e., with probability $\frac{1}{2} \pm \epsilon$) to provide a key-independent linear approximation of a cipher portion
2. Use the approximation in place of all-but one rounds of the cipher recover the last round key

S-box Approximation

- We need to check how well linear relations "fit" an S-Box
- How many lin. rel. between S-Box inputs and outputs can be written?
  - Linear relation $\Rightarrow$ polynomials with degree 1 (e.g.: $X_1 \oplus X_3 \oplus X_4$)
  - The relations can be written as $\bigoplus_i X_i = \bigoplus_j Y_j \oplus c$, $c \in \{0, 1\}$
  - Up to 8 variables involved in each relation (4 inputs, 4 outputs), two possible results $\Rightarrow 2^4 \cdot 2^4 \cdot 2 = 2^9$ possible relations
- If a given relation holds with probability $\frac{1}{2}+\epsilon$ with c=0, then same relation with c=1 yields one holding with bias $-\epsilon$
- Since we are interested in the magnitude $|\epsilon|$ of ϵ, we will arbitrarily pick c=0

Bias computations for a 4-to-4 bit S-box

- A linear input-output *relation* for a 4-to-4 bit S-box is:

$$a_0X_0 \oplus a_1X_1 \oplus a_2X_2 \oplus a_3X_3 = b_0Y_0 \oplus b_1Y_1 \oplus b_2Y_2 \oplus b_3Y_3; \quad a_i, b_i \in \{0,1\}$$

- For each choice of the binary coefficients a_i, b_i out of $(2^4)^2$=256 possible choices: $\{\langle 0000, 0000\rangle, \ldots, \langle 0010, 1001\rangle, \ldots\}$, we select a *relation*
  - Initialize two values: `ctr`$\leftarrow 0$, const `Sbox_size`=16
  - For each of the 16 S-box inputs $(x_0, x_1, x_2, x_3) \in \{0000, 0001, \ldots\}$
    - Compute the output values $(y_0, y_1, y_2, y_3) =$ `Sbox`(x_0, x_1, x_2, x_3)
    - If the relation for the given choice of a_i, b_i holds increment `ctr`
  - Store the bias value $\epsilon = \Pr - \frac{1}{2} = \frac{\texttt{ctr}}{16} - \frac{1}{2}$ for the selected relation as "`ctr` $-$ $\frac{\texttt{Sbox\_size}}{2}$" (to handle integers instead of fractions)
- Obviously, the relation given by $\langle (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3)\rangle = \langle 0000, 0000\rangle$ will always be true (sum of no input bits = sum of no output bits)

  i.e., $Pr(r)$=1 or bias $\epsilon = \frac{1}{2}$ (maximum) or *integer bias*$= \frac{\texttt{Sbox\_size}}{2}$ (maximum)

S-Box Description

Small, yet troublesome

- We will tackle a 4×4-bit S-Box, corresponding to a $\{0,1\}^4 \mapsto \{0,1\}^4$ nonlinear map
- It is obtained considering the first S-Box of DES and setting to 00 the two leftmost input bits
- The complete description is provided below (input on the top row, $X_0X_1X_2X_3$, in decimal; output nibble, $Y_0Y_1Y_2Y_3$, on the bottom one)

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
14	4	13	1	2	15	11	8	3	10	6	12	5	9	0	7

Bias table for the S-box

$$\overbrace{a_0X_0 \oplus a_1X_1 \oplus a_2X_2 \oplus a_3X_3}^{a=(a_0a_1a_2a_3)\ :\ \mathcal{X}(X)} = \overbrace{b_0Y_0 \oplus b_1Y_1 \oplus b_2Y_2 \oplus b_3Y_3}^{b=(b_0b_1b_2b_3)\ :\ \mathcal{Y}(Y)}; \quad a_i, b_i \in \{0,1\}$$

$a:\mathcal{X}(X)\downarrow$ $b:\mathcal{Y}(Y)\rightarrow$	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0 : 0	**8**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1 : X_3	0	0	**-2**	**-2**	0	0	**-2**	**6**	**2**	**2**	0	0	**2**	**2**	0	0
2 : X_2	0	0	**-2**	**-2**	0	0	**-2**	**-2**	0	0	**2**	**2**	0	0	**-6**	2
3 : $X_2 \oplus X_3$	0	0	0	0	0	0	0	0	**2**	**-6**	**-2**	**-2**	**2**	**2**	**-2**	**-2**
4 : X_1	0	**2**	0	**-2**	**-2**	**-4**	**-2**	0	0	**-2**	0	**2**	**2**	**-4**	**2**	0
5 : $X_1 \oplus X_3$	0	**-2**	**-2**	0	**-2**	0	**4**	**2**	**-2**	0	**-4**	**2**	0	**-2**	**-2**	0
6 : $X_1 \oplus X_2$	0	**2**	**-2**	**4**	**2**	0	0	**2**	0	**-2**	**2**	**4**	**-2**	0	0	**-2**
7 : $X_1 \oplus X_2 \oplus X_3$	0	**-2**	0	**2**	**2**	**-4**	**2**	0	**-2**	0	**2**	0	**4**	**2**	0	**2**
8 : X_0	0	0	0	0	0	0	0	0	**-2**	**2**	**2**	**-2**	**2**	**-2**	**-2**	**6**
9 : $X_0 \oplus X_3$	0	0	**-2**	**-2**	0	0	**-2**	**-2**	**-4**	0	**-2**	**2**	0	**4**	**2**	**-2**
A : $X_0 \oplus X_2$	0	**4**	**-2**	**2**	**-4**	0	**2**	**-2**	**2**	**2**	0	0	**2**	**2**	0	0
B : $X_0 \oplus X_2 \oplus X_3$	0	4	0	**-4**	**4**	0	**4**	0	0	0	0	0	0	0	0	0
C : $X_0 \oplus X_1$	0	**-2**	**4**	**-2**	**-2**	0	**2**	0	**2**	0	**2**	**4**	0	**2**	0	**-2**
D : $X_0 \oplus X_1 \oplus X_3$	0	**2**	**2**	0	**-2**	**4**	0	**2**	**-4**	**-2**	**2**	0	**2**	0	0	**2**
E : $X_0 \oplus X_1 \oplus X_2$	0	**2**	**2**	0	**-2**	**-4**	0	**2**	**-2**	0	0	**-2**	**-4**	**2**	**-2**	0
F : $X_0 \oplus \ldots \oplus X_3$	0	**-2**	**-4**	**-2**	**-2**	0	**2**	0	0	**-2**	**4**	**-2**	**-2**	0	**2**	0

How to read a bias table

- The previous bias table can be read as follows:
  - Choose input bits $\rightarrow$ represent them as a vector $(a_0, a_1, a_2, a_3) \rightarrow$ read them as a 4-bit binary row index value
  - Choose output bits $\rightarrow$ represent them as a vector $(b_0, b_1, b_2, b_3) \rightarrow$ read them as a 4-bit binary column index value
  - Look up the value v. If signs are not used, then a Color-Coding is adopted: red is positive, whereas blue is negative
  - The bias for the chosen relation is $\epsilon = \frac{v}{(\texttt{Sbox\_size})}$
- Example:
  - Choose input bits $\cancel{X_0}, \cancel{X_1}, \cancel{X_2}, X_3 \rightarrow (0, 0, 0, 1) \rightarrow$ row 1
  - Choose output bits $\cancel{Y_0}, Y_1, Y_2, Y_3 \rightarrow (0, 1, 1, 1) \rightarrow$ column 7
  - The value v is $+6$, thus the relation $r : X_3 = Y_1 \oplus Y_2 \oplus Y_3$ holds with probability $\Pr(r) = \frac{1}{2} + \frac{(+6)}{16} = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$
- The last input bit of the S-Box input is equal to the sum of the last three output bits ($r : X_3 = Y_1 \oplus Y_2 \oplus Y_3$) 7 times out of 8

Tackling a *Not so Trivial Cipher*

Recap

- Known: we have a set of *approximate linear relations* which can replace the S-box with a known probability
- Known: the *linear portion of the cipher* is described by linear relations
- Goal-1: Obtain a key-independent linear relation equivalent to all the rounds of the cipher save for the last one ,holding with a given bias
- Goal-2: Use the approximation to derive the final round-key through solving the eq. system

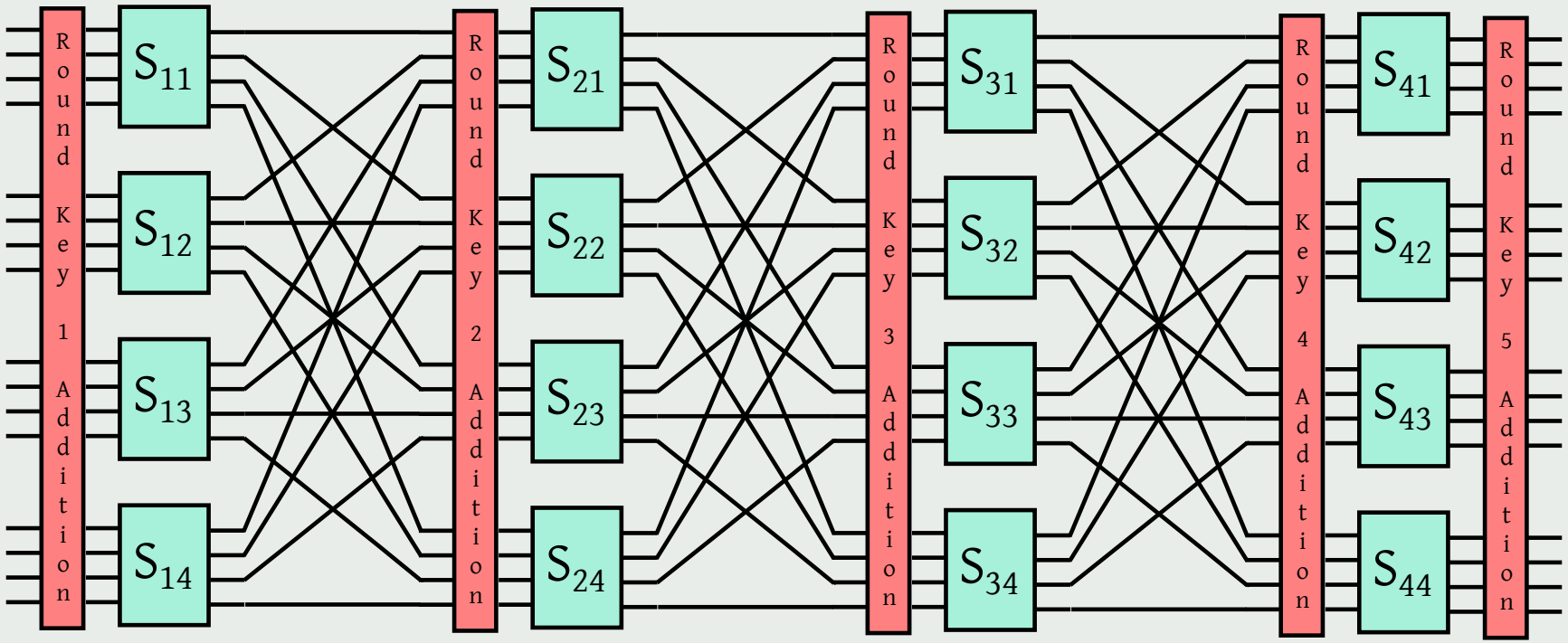
Target cipher

the *Simple* cipher

- 16-bit block, 4 rounds, 80 bit key
- No keyschedule, 16-bit used in each round
- SPN design: [AddRoundKey, SBox, Perm]$\times$3, [AddRoundKey, SBox]$\times$1, plus a final AddRoundKey
- The S-boxes are all the same, and match the one we just analysed
- Permutation: i-th bit of j-th box goes into the j-th bit of i-th Sbox
- No permutation in the last round: trivial to eliminate, acts bitwise
- The round key addition is performed via bitwise `xor`

Target cipher

the *Simple* cipher



Chaining Approximations

How to build a whole cipher approximation

- We now know how
  - to approximate a single S-box and
  - to write a linear relation for a diffusion stage
- To obtain a full cipher approximation we still need to:
  1. Find a way to compute the bias if we use more than one approximation (i.e., chain them to approximate more than a single S-Box)
  2. Cope with the effects of the intermediate key additions
- We will start with the statistical tool which allows us to combine two approximations: the Pile-up Lemma [1]

Pile-up Lemma [1] - Toolbox

- Consider two random variables Z_1, Z_2 over $\{0,1\}$, with their respective distributions

$$\Pr(Z_1{=}z_1){=}\begin{cases} p_1 & \text{if } z_1 = 0 \\ 1-p_1 & \text{if } z_1 = 1 \end{cases} \qquad \Pr(Z_2{=}z_2){=}\begin{cases} p_2 & \text{if } z_2 = 0 \\ 1-p_2 & \text{if } z_2 = 1 \end{cases}$$

- If we take Z_1 to be **independent** from Z_2, we obtain:

$$\Pr(Z_1 = z_1, Z_2 = z_2) = \begin{cases} p_1 p_2 & z_1 = 0, z_2 = 0 \\ p_1(1-p_2) & z_1 = 0, z_2 = 1 \\ (1-p_1)p_2 & z_1 = 1, z_2 = 0 \\ (1-p_1)(1-p_2) & z_1 = 1, z_2 = 1 \end{cases}$$

- We can now derive $\Pr(Z_1 \oplus Z_2{=}0)$ as
  $\Pr(Z_1 = 0, Z_2 = 0) + \Pr(Z_1 = 1, Z_2 = 1){=}p_1 p_2 + (1-p_1)(1-p_2)$

Pile-up Lemma - Toolbox

- For the sake of clarity, explicit ϵ_1, ϵ_2 rewriting: $p_1{=}\frac{1}{2}{+}\epsilon_1, p_2{=}\frac{1}{2}{+}\epsilon_2$
- The biases ϵ_1, ϵ_2, are in the range $-\frac{1}{2} \leq \epsilon_1, \epsilon_2 \leq \frac{1}{2}$ and highlight how much the probabilities deviate from an fair coin toss
- We can now obtain the following definition for $\Pr(Z_1 \oplus Z_2{=}0)$:

$$\Pr(Z_1 \oplus Z_2 = 0) = p_1 p_2 + (1 - p_1)(1 - p_2)$$

$=$

$$\frac{1}{2} + 2\epsilon_1\epsilon_2$$

- Thus, denote the bias associated to the $r : Z_1 \oplus Z_2{=}0$ as $2\epsilon_1\epsilon_2{=}\epsilon_{1,2}$

Pile-up Lemma [1]

- Following the previous results, we can generalize to $n \geq 2$ **independent** random variables, by induction:

$$\Pr(Z_1 \oplus Z_2 \oplus \dots Z_n = 0) = \frac{1}{2} + 2^{n-1} \prod_{i=1}^{n} \epsilon_i$$

- Equivalently, but in a shorter form: $\epsilon_{1,\dots,n} = 2^{n-1} \prod_{i=1}^{n} \epsilon_i$

Proof:

- Assume validity of the Lemma for $n-1$ variables: $\Pr(\bigoplus_{i=1}^{n-1} Z_i = 0) = \frac{1}{2} + \epsilon_{1,\dots,n-1}$ where $\epsilon_{1,\dots,n-1} = 2^{n-2} \prod_{i=1}^{n-1} \epsilon_i$
- Define the binary variable $W = \bigoplus_{i=1}^{n-1} Z_i$. Then $\Pr(W \oplus Z_n = 0) = \dots = \frac{1}{2} + 2\epsilon_{1,\dots,n-1} \cdot \epsilon_n$, $\Rightarrow \epsilon_{1,\dots,n} = 2^{n-1} \prod_{i=1}^{n} \epsilon_i$

Pile-up Lemma and Independence

Issues when variables are not independent

Let Z_1, Z_2, Z_3 be independent binary random variable with biases $\epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{1}{4}$. We can compute directly

$$
\begin{aligned}
Z_{12} = Z_1 \oplus Z_2 \quad &\text{with bias} \quad \epsilon_{12} = 2\epsilon_1\epsilon_2 = \tfrac{1}{8} \\
Z_{23} = Z_2 \oplus Z_3 \quad &\text{with bias} \quad \epsilon_{23} = 2\epsilon_2\epsilon_3 = \tfrac{1}{8} \\
Z_{13} = Z_1 \oplus Z_3 \quad &\text{with bias} \quad \epsilon_{13} = 2\epsilon_1\epsilon_3 = \tfrac{1}{8}
\end{aligned}
$$

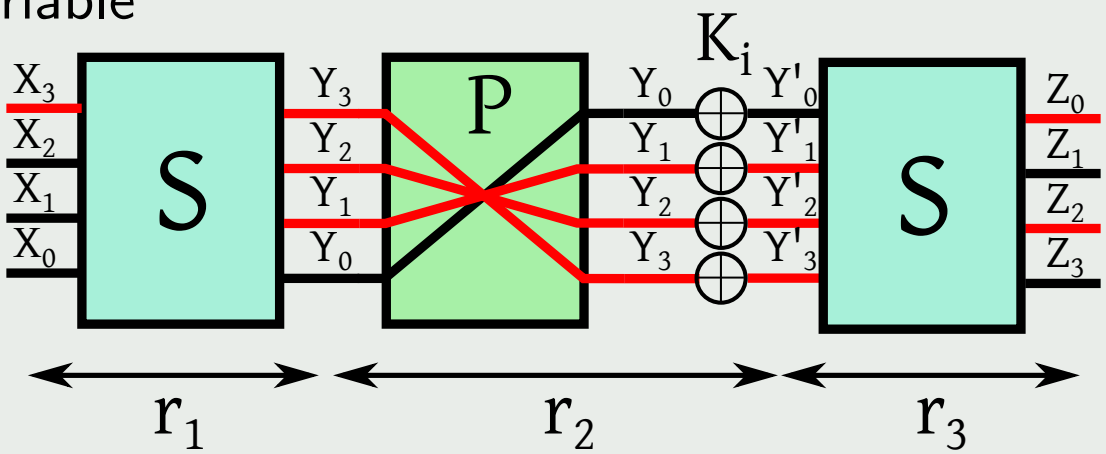
Note that Z_{12} and Z_{23} cannot be independent.

If they were independent, then by the Pile-up lemma the bias of $Z_{13} = Z_1 \oplus Z_3 = (Z_1 \oplus Z_2) \oplus (Z_2 \oplus Z_3) = Z_{12} \oplus Z_{23}$ would be equal to $2 \cdot \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{32}$, which is not the case

Use of the Pile-up lemma for combining relations

Example

Now we assume relations among input (X_i) and output (Z_i) values as equalities to 0, thus each of them can be interpreted as a random variable. The goal is to have a relation between X_i and Z_i without any other intermediate variable



$r_1 : X_3 \oplus Y_1 \oplus Y_2 \oplus Y_3 = 0$ $\quad \epsilon_1 = +\frac{6}{16}$

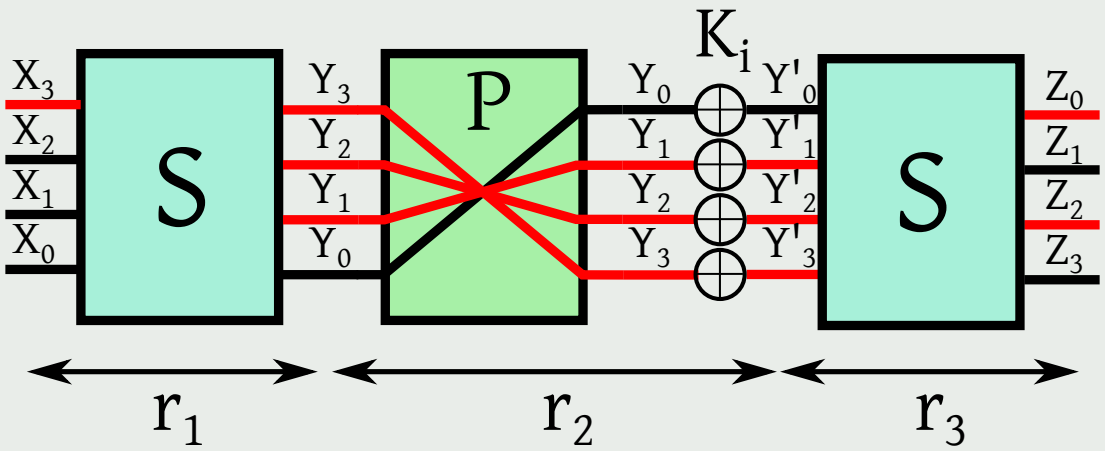
$r_2 : (Y'_3 \oplus Y_3 \oplus K_3) \oplus (Y'_1 \oplus Y_1 \oplus K_1) \oplus (Y'_2 \oplus Y_2 \oplus K_2) = 0$ $\quad \epsilon_2 = +\frac{1}{2}$

$r_3 : Y'_1 \oplus Y'_2 \oplus Y'_3 \oplus Z_0 \oplus Z_2 = 0$ $\quad \epsilon_3 = -\frac{4}{16}$

(r_2 is always true as permutation and key addition are linear)

Use of the Pile-up lemma for combining relations

Example



Combine r_2, r_3 into r_{23} adding relations memberwise and compute ϵ_{23}

$r_{23} : Y_3 \oplus Y_2 \oplus Y_1 \oplus K_1 \oplus K_2 \oplus K_3 \oplus Z_2 \oplus Z_0 = 0 \quad \epsilon_{23} = \epsilon_3 = -\frac{4}{16}$

$r_1 : \; X_3 \oplus Y_3 \oplus Y_2 \oplus Y_1 = 0 \quad \epsilon_1 = +\frac{6}{16}$

Combine r_{23} and r_1 (use pile-up lemma, assuming r_1, r_{23} as independent – this is sensible in most cases [1])

$r_{123} : X_3 \oplus Z_2 \oplus Z_1 \oplus K_0 \oplus K_1 \oplus K_2 = 0 \quad \epsilon_{123} = 2\epsilon_1\epsilon_{23} = -\frac{3}{16}$

Whole cipher approximation

- We are closing in on our target: obtain an approximation of a part of the cipher up to before the last key-addition
- We still need to do two things
  1. Choose which biases to use for the S-Boxes and which S-Boxes to approximate
  2. Deal with the four round-keys being added in between
- We will start choosing which biased relations to use and which S-Boxes to approximate
- The relations are chosen with two (possibly contrasting) goals:
  1. Minimize the number of output bits so to minimize the number of active boxes (or the bias will thin down...)
  2. Employ SBox linear approximations with the highest possible biases

Linear Path

- Notation:
  - $U_{i,j}$ represents the j-th input bit (counting from 1 to 16, bottom to top in our picture) to the i-th round S-Boxes
  - $V_{i,j}$ represents the j-th output bit (counting from 1 to 16, bottom to top in our picture) to the i-th round S-Boxes
  - $K_{i,j}$ represents the j-th bit (counting from 1 to 16, bottom to top in our picture) of the i-th round key
- We start by considering the input to the first round S-boxes: $\forall j \in [1, 16]\ U_{1,j} = P_j \oplus K_{1,j}$ as the first AddRoundKey acts on all bits
- We now choose the bias for our first step:

$$r_1 : X_0 \oplus X_2 \oplus X_3 = Y_1 \quad \text{on S-Box } S_{13}$$

$$\Pr(r_1) = 12/16, \epsilon_{r_1} = \frac{1}{4}$$

Linear Path

- Applying $r_1 : X_0 \oplus X_2 \oplus X_3 = Y_1$ ($\Pr(r_1) = 12/16, \epsilon_{r_1} = \frac{1}{4}$) on S-box S_{13}, allows us to state that

$$V_{1,6} = U_{1,5} \oplus U_{1,7} \oplus U_{1,8} = (P_5 \oplus K_{1,5}) \oplus (P_7 \oplus K_{1,7}) \oplus (P_8 \oplus K_{1,8})$$

  with probability $\Pr = \frac{1}{2} + \epsilon_{r_1} = \frac{3}{4}$ *...One box down, three to go :-)*
- Since the $V_{1,6}$ bit goes through the second AddRoundKey, $U_{2,6} = V_{1,6} \oplus K_{2,6}$
- Following the wires, we obtain that our next relation to be considered needs to involve only $U_{2,6}$ among the input bits
- We need to choose a high-bias relation involving X_1 (the second Sbox input – corresp. to $V_{1,6}$); let's take

$$r_2 : X_1 = Y_1 \oplus Y_3 \quad \text{with bias } \epsilon_{r_2} = -\frac{1}{4}$$

Linear Path

- Applying $r_2 : X_1 = Y_1 \oplus Y_3$ $(\epsilon_{r_2} = -\frac{1}{4})$ on SBox S_{23}, allows us to state that

$$V_{1,6} \oplus K_{2,6} = U_{2,6} = V_{2,6} \oplus V_{2,8}$$

  with probability $\Pr = \frac{1}{2} + \epsilon_{r_2} = \frac{1}{4}$
- ...*It's time to merge the two relations!* Applying the Pile-up lemma, we substitute $V_{1,6}$ with $P_5 \oplus K_{1,5} \oplus (P_7 \oplus K_{1,7}) \oplus P_8 \oplus K_{1,8}$, obtaining

$$r_{12} : P_5 \oplus K_{1,5} \oplus (P_7 \oplus K_{1,7}) \oplus P_8 \oplus K_{1,8} \oplus K_{2,6} = V_{2,6} \oplus V_{2,8}$$

  with a bias of $\epsilon_{r_1,r_2} = 2\epsilon_{r_1}\epsilon_{r_2} = \frac{1}{4}(-\frac{1}{4}) = -\frac{1}{8}$
- ...*Two boxes down*: we have a linear approximation in the plaintext (and key bits) up to after the second S-box layer holding with $\Pr(r_{12}) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

Linear Path

- As the previous relation involves wires belonging to two bytes of the output layer, we now need to combine two relations to approximate S_{31} and S_{33}
- Employing the same r_2 as before on both boxes we obtain that

$$V_{3,6} \oplus V_{3,8} = V_{2,6} \oplus K_{3,6} \qquad V_{3,14} \oplus V_{3,16} = V_{2,8} \oplus K_{3,14}$$

both holding with probability $-\frac{1}{4}$, each

- Piling them up we obtain that

$$r_3 : V_{3,6} \oplus V_{3,8} \oplus V_{2,6} \oplus K_{3,6} \oplus V_{3,14} \oplus V_{3,16} \oplus V_{2,8} \oplus K_{3,14} = 0$$

holding with a (piled-up) bias of $2(-\frac{1}{4})(-\frac{1}{4}) = \frac{1}{8}$

Linear Path

- Combining our approximation of the third layer of boxes

$$r_3 : V_{3,6} \oplus V_{3,8} \oplus V_{2,6} \oplus K_{3,6} \oplus V_{3,14} \oplus V_{3,16} \oplus V_{2,8} \oplus K_{3,14} = 0$$

$(Pr(r_3) = \frac{1}{2} + \frac{1}{8})$ with the previous one for the first two

$$r_{12} : V_{2,6} \oplus V_{2,8} = P_5 \oplus K_{1,5} \oplus (P_7 \oplus K_{1,7}) \oplus P_8 \oplus K_{1,8} \oplus K_{2,6}$$

$(Pr(r_{12}) = \frac{1}{2} - \frac{1}{8})$

$$r_{123} : V_{3,6} \oplus V_{3,8} \oplus V_{3,14} \oplus V_{3,16} \oplus P_5 \oplus P_7 \oplus P_8 \oplus \Sigma_k = 0$$

holding with bias $2(\frac{1}{8})(-\frac{1}{8}) = -\frac{1}{32}$, where Σ_k is

$$\Sigma_k = K_{3,14} \oplus K_{3,6} \oplus K_{1,5} \oplus K_{1,7} \oplus K_{1,8} \oplus K_{2,6}$$

Dropping the key bits

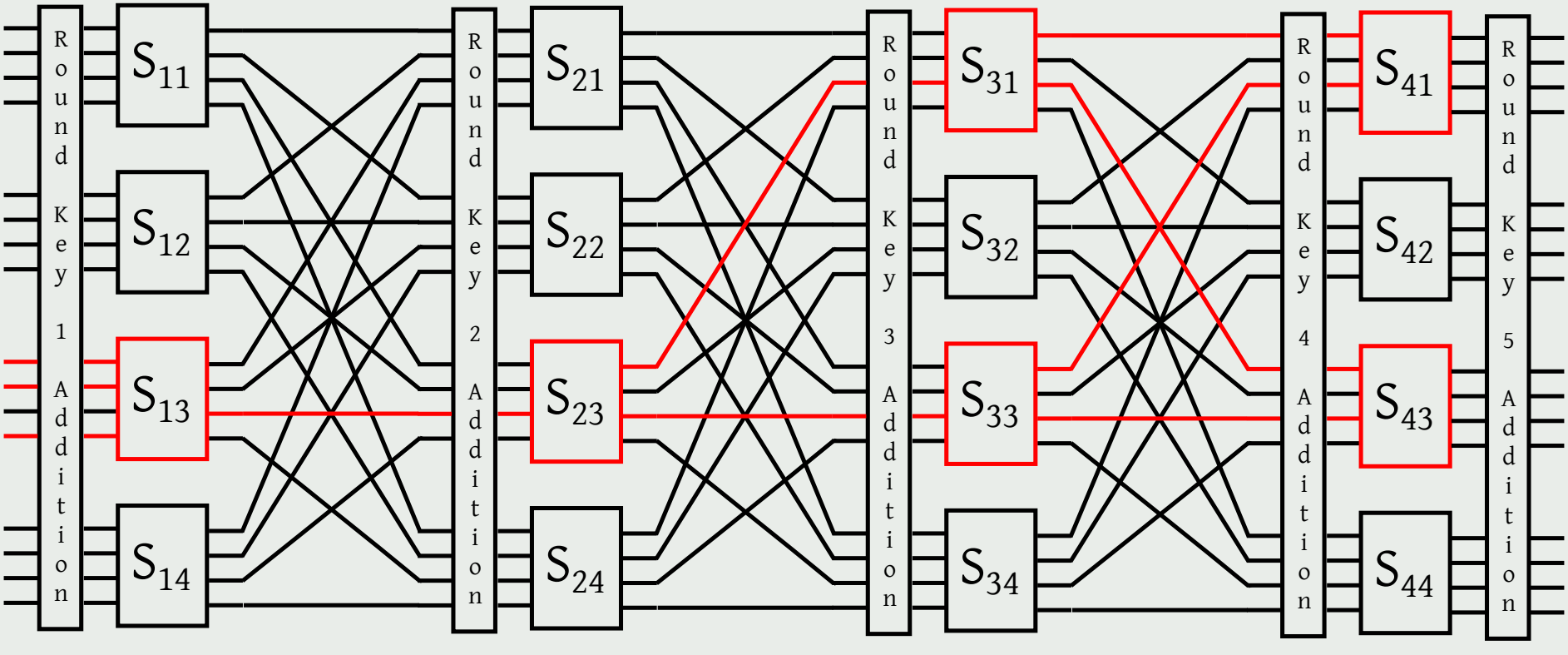
- Adding the fourth round key, and packing all the key bits in Σ_k we get

$$r_4 : U_{4,6} \oplus U_{4,8} \oplus U_{4,14} \oplus U_{4,16} \oplus P_5 \oplus P_7 \oplus P_8 \oplus \Sigma_k = 0$$

  holding with a bias of $-\frac{1}{32}$
- We now consider the sum of the involved key bits as a single value Σ_k: this can only assume 0 or 1 as its value
- **As the key is fixed, all the key bits have (at most) the net effect of flipping the sign of the relation bias which is thus $\pm\frac{1}{32}$**
- There we go :) We have a linear approximation, not dependent on the key bits, from the plaintext to the input of the fourth SBox layer!

Linear Cryptanalysis

Complete linear approximation path



Linear Cryptanalysis

Extracting the key bits for the last round

Given a linear path approximation of the cipher, r, involving some ptx bits and some bits coming from the penultimate key-addition layer, holding with bias ϵ

- Call *partial subkeys* the key bits of the last round-key influenced by the output bits of our approximation
- Collect $N \geq \frac{1}{\epsilon^2}$ ptx/ctx pairs [1]

1. For each given partial subkey value, sk_i, count how many ptx/ctx pairs make the relation r hold (say N_{sk_i})
2. Compute the sample probabilities $p_i = \frac{N_{sk_i}}{N}$ for each sk_i value
3. Check which value $p_i = \frac{1}{2} \pm \epsilon \Rightarrow sk_i$ is a candidate subkey!

Lather, rinse, repeat with other linear approximations to extract the other partial subkeys

Linear Cryptanalysis

Summary

1. Analyze the cipher Sboxes, computing the linear bias table
2. Choose viable relations for an S-Box
3. Propagate the biases up to the last-but-one AddRoundKey employing the Pile-up lemma
4. Collect $\geq \frac{1}{\epsilon^2}$ ptx/ctx pairs
5. Check for all the possible partial subkeys involved in the linear approximation which one makes it hold with the correct bias
6. Repeat for other subkeys, eventually removing rounds until all the key material is found

Further readings

Practice and real world examples

- A `C` implementation of the whole attack and a computer for the linear biases is available on the website
- A real world cipher known to be terribly vulnerable to linear cryptanalysis is FEAL [2]:
  - it's only 4 rounds, so you can try and crack it
- For the more determined: you can read [1] to explore the full DES linear cryptanalysis

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