# Block Cipher Cryptanalysis - Part 1

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# Overview

## Lesson contents

- Basics of Block Cipher Cryptanalysis
- Algebraic Known Plaintext Attack of Block Ciphers w/o Sboxes
  - The *Trivial* cipher
  - Algebraic Cryptanalysis of *Trivial*
- Statistical Known Plaintext Attack: Linear Cryptanalysis
  - The *Simple* cipher
  - Linear Cryptanalysis of *Simple*

# Block Cipher Cryptanalysis

## Approaches

- Modern block cipher cryptanalysis aims at either recovering the key with an effort smaller than bruteforce
- *Known-ciphertext-only* cryptanalysis is a symptom of a deeply flawed cipher (Caesar, Vigenére, Playfair *et al.*)
- We will tackle techniques requiring either a *known plaintext* or a *related (chosen) plaintext* assumption
- Block cipher cryptanalyses are usually split into two families: algebraic and statistical

## Algebraic Analysis

A cipher can be seen as a *large* set of Boolean simultaneous equations involving key bits (variables), ptx and ctx bits (known terms).

Either one or both of the following findings should be obtained efficiently to have a successful cryptanalysis

- A solution (in closed form), for the simultaneous equations yields the key from one (or more) ptx/ctx pairs
- The discovery that only a subset of the key bits are actually involved in the simultaneous equations, while a portion of them self-cancel
  - Only the involved key bits are effectively used by the cipher in encryption

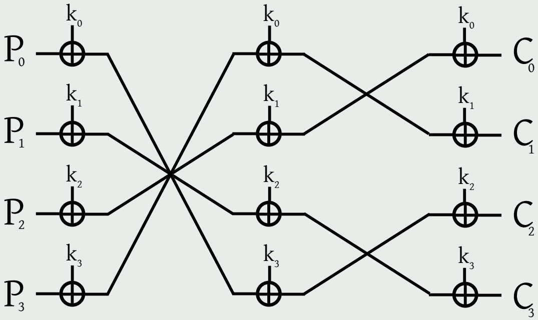
## Statistical Analysis

- A perfect cipher is be able to produce a ctx independent from the ptx
- All the computationally secure ones produce a ctx where the probability that a bit is either one or zero is biased ($= \frac{1}{2} \pm \epsilon$) by the ptx value
- Thus, a relation $r$ between ptx and ctx bits may hold with bias $\epsilon \neq 0$
  - E.g., "even numbers of 1s in the ptx $\rightarrow$ even number of 1s in the ctx"
- Goal: find such relations between ptx and ctx, and exploit the statistical biases to extract a portion of the key

## Starting out

- From now on, *linear* will be referring to linearity w.r.t. the addition modulo 2, i.e., eXclusive-OR (`xor`)
- Recall that the multiplication mod 2 is the Boolean "`and`" operation and that $x^2 \bmod 2 = x$ if you only care for solutions in $\{0, 1\}$
- First step: observe that any linear cipher is trivial to cryptanalyze
  - Obtaining the key with a *known plaintext attack* = solving a set of simultaneous linear equations
- First example: the *Trivial* cipher, only has *diffusion* and *key addition*

## the "Trivial" Cipher




- 4-bit block, 4-bit key
- Key schedule: a repetition of the key

## Cryptanalysis of "Trivial": Algebraic Approach

- Rewrite the relations between ptx, ctx and key bits as equations with binary coefficients:

$$
\begin{aligned}
C_0 &= P_2 \oplus k_2 \oplus k_1 \oplus k_0 \\
C_1 &= P_3 \oplus k_3 \oplus k_0 \oplus k_1 \\
C_2 &= P_0 \oplus k_0 \oplus k_3 \oplus k_2 \\
C_3 &= P_1 \oplus k_1 \oplus k_2 \oplus k_3
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
P_2 \oplus C_0 &= k_2 \oplus k_1 \oplus k_0 \\
P_3 \oplus C_1 &= k_3 \oplus k_0 \oplus k_1 \\
P_0 \oplus C_2 &= k_0 \oplus k_3 \oplus k_2 \\
P_1 \oplus C_3 &= k_1 \oplus k_2 \oplus k_3
\end{aligned}
$$

- This is a set of linear simultaneous eq.s, where the ptxs and ctxs are known: just solve it for $k_0, k_1, k_2, k_3$
- Any cipher made only with a linear *key addition* and a linear *diffusion* layer can be broken with the same technique

## Cryptanalysis of "Trivial": Statistical Approach

- Since in *Trivial* $C_0 = P_2 \oplus k_2 \oplus k_1 \oplus k_0$:
  - If the relation $\mathbf{r} : \mathbf{C_0} = \mathbf{P_2}$ holds, then either:
    1. $k_0 = k_2 = k_1 = 0$
    2. $k_0 = k_1 = 1, k_2 = 0$
    3. $k_0 = k_2 = 1, k_1 = 0$
    4. $k_1 = k_2 = 1, k_0 = 0$
  - If the relation $\mathbf{r}' : \mathbf{C_0} \neq \mathbf{P_2}$ holds, then either:
    1. $k_0 = k_2 = k_1 = 1$
    2. $k_0 = k_1 = 0, k_2 = 1$
    3. $k_0 = k_2 = 0, k_1 = 1$
    4. $k_1 = k_2 = 0, k_0 = 1$
- In both cases, looking at the value of a ptx-ctx bit pair, out of $2^3 = 8$ possible values for $k_0, k_1, k_2$ only 4 possibilities are left
- This is equivalent to a keyspace reduction by a factor of 2, and can always be done since either $r$ holds with $Pr(r) = 1$, or $r'$ does.

# Nonlinear Relations

## Enter the S-Boxes

- Sound ciphers have nonlinear components (S-Boxes) preventing efficient closed form solution of simultaneous equations
  - Solving simultaneous equations with deg. $> 1$ in $\mathbb{Z}_p$ is NP-Complete
- Without S-Boxes, the linear relations between ptx and ctx bits always hold (i.e., with probability $=1$)
- With S-Boxes, linear relations may still hold with a probability $\neq \frac{1}{2}$

## Key Idea of Linear Cryptanalysis

1. Exploit the relations holding with a bias (i.e., with probability $\frac{1}{2} \pm \epsilon$) to provide a key-independent linear approximation of a cipher portion
2. *(i)* Use the approximation in place of all-but one rounds of the cipher, and then *(ii)* use it to recover the last round key

## Approximation of a 4-to-4 S-box

- We need to check how well linear relations "fit" the S-Box
- How many linear eq.s between S-Box inputs and outputs can be written?
  - Linear equation $\Rightarrow$ a polynomial with degree 1 (e.g.: $X_1 \oplus X_3 \oplus X_4$)
  - The equations can be written as $\bigoplus_i X_i = \bigoplus_j Y_j \oplus c$, $c \in \{0, 1\}$
  - Up to 8 variables involved in each relation (4 inputs, 4 outputs), two possible results $\Rightarrow 2^4 \cdot 2^4 \cdot 2 = 2^9$ possible relations
- If a given eq. with $c$=0 holds with probability $\frac{1}{2}+\epsilon$, then same eq. with $c$=1 would hold with probability $\frac{1}{2} - \epsilon$
- Since we are interested in the magnitude $|\epsilon|$ of $\epsilon$, we will arbitrarily pick $c$=0

# Linear Cryptanalysis

## Bias computations for a 4-to-4 bit S-box

- A linear input-output *relation* for a 4-to-4 bit S-box is:
  $a_0X_0 \oplus a_1X_1 \oplus a_2X_2 \oplus a_3X_3 = b_0Y_0 \oplus b_1Y_1 \oplus b_2Y_2 \oplus b_3Y_3; \quad a_i, b_i \in \{0,1\}$
- For each choice of the binary coefficients $a_i, b_i$ out of $(2^4)^2{=}256$ possible choices $\{\langle 0000, 0000\rangle, \ldots, \langle 0010, 1001\rangle, \ldots\}$, a different equation can be written. To assess how well each equation approximates the S-box, we compute their probabilities as follows.
  - Given one equation (e.g., $X_0 \oplus X_3 = Y_2$), let us initialize a counter $\texttt{ctr} \leftarrow 0$
  - For each of the 16 S-box inputs $(x_0, x_1, x_2, x_3) \in \{0000, 0001, \ldots\}$
    - Compute the output values $(y_0, y_1, y_2, y_3) = \texttt{Sbox}(x_0, x_1, x_2, x_3)$
    - If the equation at hand holds, then increment $\texttt{ctr}$
  - Store the bias value $\epsilon{=}\Pr - \frac{1}{2} {=} \frac{\texttt{ctr-8}}{16}$ for the equation at hand as "$\texttt{ctr} - \frac{\texttt{Sbox\_size}}{2}$" ($\texttt{Sbox\_size}{=}16$, to handle integers and not fractions)
- Obviously, the relation given by $\langle (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3)\rangle {=} \langle 0000, 0000\rangle$ will always be true (sum of no input bits = sum of no output bits)
  i.e., $Pr(r){=}1$ or bias $\epsilon{=}\frac{1}{2}$ (maximum) or *integer bias*$=\frac{\texttt{Sbox\_size}}{2}$ (maximum)

# S-Box Description

## Small, yet troublesome

- We will tackle a 4×4-bit S-Box, corresponding to a $\{0,1\}^4 \mapsto \{0,1\}^4$ nonlinear map
- It is obtained considering the first S-Box of DES and setting to 00 the two leftmost input bits
- The complete description is provided below (input on the top row, $X_0X_1X_2X_3$, in hexadecimal; output nibble, $Y_0Y_1Y_2Y_3$, on the bottom one)

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| E | 4 | D | 1 | 2 | F | B | 8 | 3 | A | 6 | C | 5 | 9 | 0 | 7 |

# Bias table for the S-box

$$\overbrace{a_0X_0 \oplus a_1X_1 \oplus a_2X_2 \oplus a_3X_3}^{a=(a_0a_1a_2a_3)\ :\ \mathcal{X}(X)} = \overbrace{b_0Y_0 \oplus b_1Y_1 \oplus b_2Y_2 \oplus b_3Y_3}^{b=(b_0b_1b_2b_3)\ :\ \mathcal{Y}(Y)}; \quad a_i, b_i \in \{0,1\}$$

| $a : \mathcal{X}(X) \downarrow$ $b : \mathcal{Y}(Y) \rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 : 0 | **8** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 : $X_3$ | 0 | 0 | **-2** | **-2** | 0 | 0 | **-2** | **6** | **2** | **2** | 0 | 0 | **2** | **2** | 0 | 0 |
| 2 : $X_2$ | 0 | 0 | **-2** | **-2** | 0 | 0 | **-2** | **-2** | 0 | 0 | **2** | **2** | 0 | 0 | **-6** | 2 |
| 3 : $X_2 \oplus X_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **2** | **-6** | **-2** | **-2** | **2** | **2** | **-2** | **-2** |
| 4 : $X_1$ | 0 | **2** | 0 | **-2** | **-2** | **-4** | **-2** | 0 | 0 | **-2** | 0 | **2** | **2** | **-4** | **2** | 0 |
| 5 : $X_1 \oplus X_3$ | 0 | **-2** | **-2** | 0 | **-2** | 0 | **4** | **2** | **-2** | 0 | **-4** | **2** | 0 | **-2** | **-2** | 0 |
| 6 : $X_1 \oplus X_2$ | 0 | **2** | **-2** | **4** | **2** | 0 | 0 | **2** | 0 | **-2** | **2** | **4** | **-2** | 0 | 0 | **-2** |
| 7 : $X_1 \oplus X_2 \oplus X_3$ | 0 | **-2** | 0 | **2** | **2** | **-4** | **2** | 0 | **-2** | 0 | **2** | 0 | **4** | **2** | 0 | **2** |
| 8 : $X_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **-2** | **2** | **2** | **-2** | **2** | **-2** | **-2** | **6** |
| 9 : $X_0 \oplus X_3$ | 0 | 0 | **-2** | **-2** | 0 | 0 | **-2** | **-2** | **-4** | 0 | **-2** | **2** | 0 | **4** | **2** | **-2** |
| $A$ : $X_0 \oplus X_2$ | 0 | **4** | **-2** | **2** | **-4** | 0 | **2** | **-2** | **2** | **2** | 0 | 0 | **2** | **2** | 0 | 0 |
| $B$ : $X_0 \oplus X_2 \oplus X_3$ | 0 | **4** | 0 | **-4** | **4** | 0 | **4** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C$ : $X_0 \oplus X_1$ | 0 | **-2** | **4** | **-2** | **-2** | 0 | **2** | 0 | **2** | 0 | **2** | **4** | 0 | **2** | 0 | **-2** |
| $D$ : $X_0 \oplus X_1 \oplus X_3$ | 0 | **2** | **2** | 0 | **-2** | **4** | 0 | **2** | **-4** | **-2** | **2** | 0 | **2** | 0 | 0 | **2** |
| $E$ : $X_0 \oplus X_1 \oplus X_2$ | 0 | **2** | **2** | 0 | **-2** | **-4** | 0 | **2** | **-2** | 0 | 0 | **-2** | **-4** | **2** | **-2** | 0 |
| $F$ : $X_0 \oplus \ldots \oplus X_3$ | 0 | **-2** | **-4** | **-2** | **-2** | 0 | **2** | 0 | 0 | **-2** | **4** | **-2** | **-2** | 0 | **2** | 0 |

## How to read a bias table

- The previous bias table can be read as follows:
  - Choose input bits $\rightarrow$ represent them as a vector $(a_0, a_1, a_2, a_3) \rightarrow$ read them as a 4-bit binary row index value
  - Choose output bits $\rightarrow$ represent them as a vector $(b_0, b_1, b_2, b_3) \rightarrow$ read them as a 4-bit binary column index value
  - Look up the value $v$. If signs are not used, then a Color-Coding is adopted: red is positive, whereas blue is negative
  - The bias for the chosen relation is $\epsilon = \frac{v}{(\texttt{Sbox\_size})}$
- Example:
  - Choose input bits $\cancel{X_0}, \cancel{X_1}, \cancel{X_2}, X_3 \rightarrow (0, 0, 0, 1) \rightarrow$ row 1
  - Choose output bits $\cancel{Y_0}, Y_1, Y_2, Y_3 \rightarrow (0, 1, 1, 1) \rightarrow$ column 7
  - The value $v$ is $+6$, thus the relation $r : X_3 {=} Y_1 \oplus Y_2 \oplus Y_3$ holds with probability $\Pr(r) {=} \frac{1}{2} + \frac{(+6)}{16} = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$
- The last input bit of the S-Box input is equal to the sum of the last three output bits $(r : X_3 {=} Y_1 \oplus Y_2 \oplus Y_3)$ 7 times out of 8

# Tackling a *Not so Trivial Cipher*

## Recap

- Known: we have a set of *approximate linear relations* which can replace the S-box with a known probability
- Known: the *linear portion of the cipher* is described by linear relations
- Goal-1: Obtain a key-independent linear relation equivalent to all the rounds of the cipher save for the last one, holding with a given bias
- Goal-2: Use the approximation to derive a group of bits of the final round-key. The values guessed for a group of bits in input to the last round are employed together with the ctx bits to recover some bits of the last round key

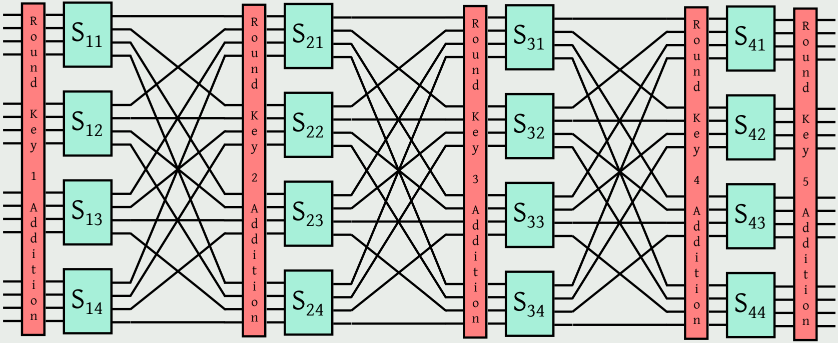
# Target cipher

## the *Simple* cipher

- 16-bit block, 4 rounds, 80-bit key
- No keyschedule, 16-bit used in each round
- SPN design: [AddRoundKey, SBox, Perm]$\times$3, [AddRoundKey, SBox]$\times$1, plus a final AddRoundKey
- The round key addition is performed via bitwise `xor`
- The S-boxes are all the same, and match the one we just analysed
- Permutation: $i$-th bit of $j$-th box goes into the $j$-th bit of $i$-th Sbox
- No permutation in the last round: trivial to eliminate as it acts bitwise; an extra round key addition is performed to secure the inputs to the last round s-boxes

# Target cipher

## the *Simple* cipher



# Chaining Approximations

## How to build a whole cipher approximation

- We now know how
  - to approximate a single S-box and
  - to write a linear relation for a diffusion stage
- To obtain a full cipher approximation we still need to:
  1. Find a way to compute the bias if we use more than one approximation (i.e., chain them to approximate more than a single S-Box)
  2. Cope with the effects of the intermediate key additions
- We will start with the statistical tool which allows us to combine two approximations: the Pile-up Lemma [1]

## Pile-up Lemma [1] - Toolbox

- Consider two random variables $Z_1, Z_2$ over $\{0, 1\}$, with their respective distributions

$$\Pr(Z_1{=}z_1){=}\begin{cases} p_1 & \text{if } z_1 = 0 \\ 1 - p_1 & \text{if } z_1 = 1 \end{cases} \qquad \Pr(Z_2{=}z_2){=}\begin{cases} p_2 & \text{if } z_2 = 0 \\ 1 - p_2 & \text{if } z_2 = 1 \end{cases}$$

- If we take $Z_1$ to be **independent** from $Z_2$, we obtain:

$$\Pr(Z_1 = z_1, Z_2 = z_2) = \begin{cases} p_1 p_2 & z_1 = 0, z_2 = 0 \\ p_1(1 - p_2) & z_1 = 0, z_2 = 1 \\ (1 - p_1)p_2 & z_1 = 1, z_2 = 0 \\ (1 - p_1)(1 - p_2) & z_1 = 1, z_2 = 1 \end{cases}$$

- We can now derive $\Pr(Z_1 \oplus Z_2{=}0)$ as $\Pr(Z_1 = 0, Z_2 = 0) + \Pr(Z_1 = 1, Z_2 = 1){=}p_1 p_2{+}(1 - p_1)(1 - p_2)$

## Pile-up Lemma - Toolbox

- For the sake of clarity, explicit $\epsilon_1, \epsilon_2$ rewriting: $p_1{=}\frac{1}{2}{+}\epsilon_1, p_2{=}\frac{1}{2}{+}\epsilon_2$
- The biases $\epsilon_1, \epsilon_2$, are in the range $-\frac{1}{2} \leq \epsilon_1, \epsilon_2 \leq \frac{1}{2}$ and highlight how much the probabilities deviate from an fair coin toss
- We can now obtain the following definition for $\Pr(Z_1{\oplus}Z_2{=}0)$:

$$\Pr(Z_1 \oplus Z_2 = 0) = p_1 p_2 + (1 - p_1)(1 - p_2)$$

$=$

$$\frac{1}{2} + 2\epsilon_1\epsilon_2$$

- Let us denote the bias associated to the $r : Z_1 \oplus Z_2{=}0$ as $2\epsilon_1\epsilon_2{=}\epsilon_{1,2}$

## Pile-up Lemma [1]

- Following the previous results, we can generalize to $n \geq 2$ **independent** random variables, by induction:

$$\Pr(Z_1 \oplus Z_2 \oplus \dots Z_n = 0) = \frac{1}{2} + 2^{n-1} \prod_{i=1}^{n} \epsilon_i$$

- Equivalently, $\frac{1}{2} + \epsilon_{1,\dots,n}$ assuming $\epsilon_{1,\dots,n} = 2^{n-1} \prod_{i=1}^{n} \epsilon_i$

**Proof**:

- Assume validity of the Lemma for $n-1$ variables: $\Pr(\bigoplus_{i=1}^{n-1} Z_i = 0) = \frac{1}{2} + \epsilon_{1,\dots,n-1}$ where $\epsilon_{1,\dots,n-1} = 2^{n-2} \prod_{i=1}^{n-1} \epsilon_i$
- Define the binary variable $W = \bigoplus_{i=1}^{n-1} Z_i$. Then $\Pr(W \oplus Z_n = 0) = \dots = \frac{1}{2} + 2\epsilon_{1,\dots,n-1} \cdot \epsilon_n$, $\Rightarrow \epsilon_{1,\dots,n} = 2^{n-1} \prod_{i=1}^{n} \epsilon_i$

# Pile-up Lemma and Independence

## Example of issues when the random variables are not independent

Let $Z_1$, $Z_2$, $Z_3$ be independent binary random variable with biases $\epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{1}{4}$. We can compute directly

$$\begin{aligned} Z_{12} &= Z_1 \oplus Z_2 && \text{with bias} && \epsilon_{12} = 2\epsilon_1\epsilon_2 = \tfrac{1}{8} \\ Z_{23} &= Z_2 \oplus Z_3 && \text{with bias} && \epsilon_{23} = 2\epsilon_2\epsilon_3 = \tfrac{1}{8} \\ Z_{13} &= Z_1 \oplus Z_3 && \text{with bias} && \epsilon_{13} = 2\epsilon_1\epsilon_3 = \tfrac{1}{8} \end{aligned}$$

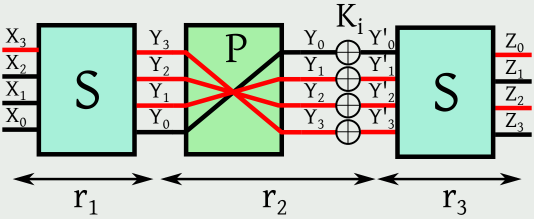
Note that the random variables $Z_{12}$ and $Z_{23}$ cannot be independent.

If they were independent, then by the Pile-up lemma the bias of $Z_{13} = Z_1 \oplus Z_3 = (Z_1 \oplus Z_2) \oplus (Z_2 \oplus Z_3) = Z_{12} \oplus Z_{23}$ would be equal to $2 \cdot \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{32}$, which is not the case

# Use of the Pile-up lemma for combining relations

## Example

Now we assume relations among input and output values of a cipher round as equalities to 0, associating each of them to a random variable $(X_i, Z_i)$. The goal is to have a relation between $X_i$ and $Z_i$ without any other intermediate variable (in particular, without the ones associated with a round key bit)



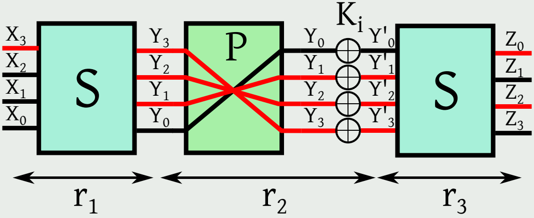

$r_1 : X_3 \oplus Y_1 \oplus Y_2 \oplus Y_3 = 0 \qquad \epsilon_1 = +\frac{6}{16}$

$r_2 : (Y_3' \oplus Y_3 \oplus K_3) \oplus (Y_1' \oplus Y_1 \oplus K_1) \oplus (Y_2' \oplus Y_2 \oplus K_2) = 0 \qquad \epsilon_2 = +\frac{1}{2}$

$r_3 : Y_1' \oplus Y_2' \oplus Y_3' \oplus Z_0 \oplus Z_2 = 0 \qquad \epsilon_3 = -\frac{4}{16}$

($r_2$ is always true (i.e., $\epsilon = +\frac{1}{2}$) as permutation and key addition are linear)

### Example




Combine $r_2, r_3$ into $r_{23}$ adding relations member-wise and compute $\epsilon_{23}$

$r_{23}: Y_3 \oplus Y_2 \oplus Y_1 \oplus Z_0 \oplus Z_2 \oplus K_3 \oplus K_2 \oplus K_1 = 0 \quad \epsilon_{23} = 2\epsilon_2\epsilon_3 = \epsilon_3 = -\frac{4}{16}$

$r_1: X_3 \oplus Y_3 \oplus Y_2 \oplus Y_1 = 0 \quad \epsilon_1 = +\frac{6}{16}$

Combine $r_{23}$ and $r_1$ (use pile-up lemma, assuming $r_1$, $r_{23}$ as independent) $r_{123}: X_3 \oplus Z_0 \oplus Z_2 = (K_3 \oplus K_2 \oplus K_1)$ we consider key bits as constants, which implies considering $r_{123}: X_3 \oplus Z_0 \oplus Z_2 = 0$, with $|\epsilon_{123}| = |2\epsilon_1\epsilon_{23}| = \frac{3}{16}$

## Whole cipher approximation

- We are closing in on our target: obtain an approximation of a part of the cipher up to the input of the last round s-boxes
- We still need to do two things
  1. Choose which biases to use for the S-Boxes and which S-Boxes to approximate
  2. Deal with the four round-keys being added in between
- We will start choosing which biased relations to use and which S-Boxes to approximate
- The relations are chosen with two (possibly contrasting) goals:
  1. Minimize the number of output bits so to minimize the number of active boxes (or the bias will thin down...)
  2. Employ SBox linear approximations with the highest possible biases

## Linear Path along the *Simple* cipher

- Notation:
  - $U_{i,j}$ represents the $j$-th input bit (counting from 1 to 16, bottom to top in our picture) to the $i$-th round S-Boxes (counting from 1 to 4)
  - $V_{i,j}$ represents the $j$-th output bit (counting from 1 to 16, bottom to top in our picture) to the $i$-th round S-Boxes
  - $K_{i,j}$ represents the $j$-th bit (counting from 1 to 16, bottom to top in our picture) of the $i$-th round key
- We start by considering the input to the first round S-boxes: $\forall j \in [1, 16]\ U_{1,j} = P_j \oplus K_{1,j}$ as the first AddRoundKey acts on all bits
- We now choose an eq. for $S_{13}$ which possibly includes a minimum number of output variables:

$$r_1 : X_0 \oplus X_2 \oplus X_3 = Y_1 \quad \text{on S-Box}\, S_{13}$$

$$\text{from the bias table we take } 4, \epsilon_{r_1} = \frac{4}{16} = \frac{1}{4}, \Pr(r_1) = \frac{3}{4}$$

## Linear Path along the *Simple* cipher

- Applying $r_1 : X_0 \oplus X_2 \oplus X_3 = Y_1$ on S-box $S_{13}$, allows us to state that

  $$V_{1,6} = U_{1,5} \oplus U_{1,7} \oplus U_{1,8} = (P_5 \oplus K_{1,5}) \oplus (P_7 \oplus K_{1,7}) \oplus (P_8 \oplus K_{1,8})$$

  with probability $\Pr(r_1) = 3/4, \epsilon_{r_1} = \frac{1}{4}$ *One box down, three to go :-)*
- Since the $V_{1,6}$ bit goes through the second AddRoundKey, $U_{2,6} = V_{1,6} \oplus K_{2,6}$
- Following the wires, we obtain that our next relation to be considered needs to involve only $U_{2,6}$ among the input bits
- We need to choose a high-bias relation involving $X_1$ (the second Sbox input – corresp. to $V_{1,6}$); let's take

  $$r_2 : X_1 = Y_1 \oplus Y_3 \quad \text{with bias } \epsilon_{r_2} = -\frac{1}{4}$$

## Linear Path along the *Simple* cipher

- Applying $r_2 : X_1 = Y_1 \oplus Y_3$ ($\epsilon_{r_2} = -\frac{1}{4}$) on SBox $S_{23}$, allows us to state that

$$V_{1,6} \oplus K_{2,6} = U_{2,6} = V_{2,6} \oplus V_{2,8}$$

  with probability $\Pr = \frac{1}{2} + \epsilon_{r_2} = \frac{1}{4}$
- ...*It's time to merge the two relations!* Applying the Pile-up lemma, we substitute $V_{1,6}$ with $P_5 \oplus K_{1,5} \oplus (P_7 \oplus K_{1,7}) \oplus P_8 \oplus K_{1,8}$, obtaining

$$r_{12} : P_5 \oplus K_{1,5} \oplus (P_7 \oplus K_{1,7}) \oplus P_8 \oplus K_{1,8} \oplus K_{2,6} = V_{2,6} \oplus V_{2,8}$$

  with a bias of $\epsilon_{r_1,r_2} = 2\epsilon_{r_1}\epsilon_{r_2} = \frac{1}{4}(-\frac{1}{4}) = -\frac{1}{8}$
- ...*Two boxes down*: we have a linear approximation in the plaintext (and key bits) up to after the second S-box layer holding with $\Pr(r_{12}) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

## Linear Path along the *Simple* cipher

- As the previous relation involves S-box outputs ($V_{2,6}$, $V_{2,8}$) wired to two different boxes on the 3rd layer, we now need to combine two relations to approximate $S_{31}$ and $S_{33}$
- Employing the same $r_2$ as before on both boxes we obtain that

$$V_{3,6} \oplus V_{3,8} = V_{2,6} \oplus K_{3,6} \qquad V_{3,14} \oplus V_{3,16} = V_{2,8} \oplus K_{3,14}$$

both holding with probability $-\frac{1}{4}$

- Piling them up (i.e., xor-ing them) we obtain that

$$r_3 : V_{3,6} \oplus V_{3,8} \oplus V_{2,6} \oplus K_{3,6} \oplus V_{3,14} \oplus V_{3,16} \oplus V_{2,8} \oplus K_{3,14} = 0$$

holding with a (piled-up) bias of $2(-\frac{1}{4})(-\frac{1}{4}) = \frac{1}{8}$

## Linear Path along the *Simple* cipher

- Combining our approximation of the third layer of boxes

$$r_3 : V_{3,6} \oplus V_{3,8} \oplus V_{2,6} \oplus K_{3,6} \oplus V_{3,14} \oplus V_{3,16} \oplus V_{2,8} \oplus K_{3,14} = 0$$

$(Pr(r_3) = \frac{1}{2} + \frac{1}{8})$ with the previous one for the first two

$$r_{12} : V_{2,6} \oplus V_{2,8} = P_5 \oplus K_{1,5} \oplus (P_7 \oplus K_{1,7}) \oplus P_8 \oplus K_{1,8} \oplus K_{2,6}$$

$(Pr(r_{12}) = \frac{1}{2} - \frac{1}{8})$ gives us the following relation

$$r_{123} : V_{3,6} \oplus V_{3,8} \oplus V_{3,14} \oplus V_{3,16} \oplus P_5 \oplus P_7 \oplus P_8 \oplus \Sigma_k = 0$$

holding with bias $2(\frac{1}{8})(-\frac{1}{8}) = -\frac{1}{32}$, where $\Sigma_k$ is

$$\Sigma_k = K_{3,14} \oplus K_{3,6} \oplus K_{1,5} \oplus K_{1,7} \oplus K_{1,8} \oplus K_{2,6}$$

## Dropping the key bits

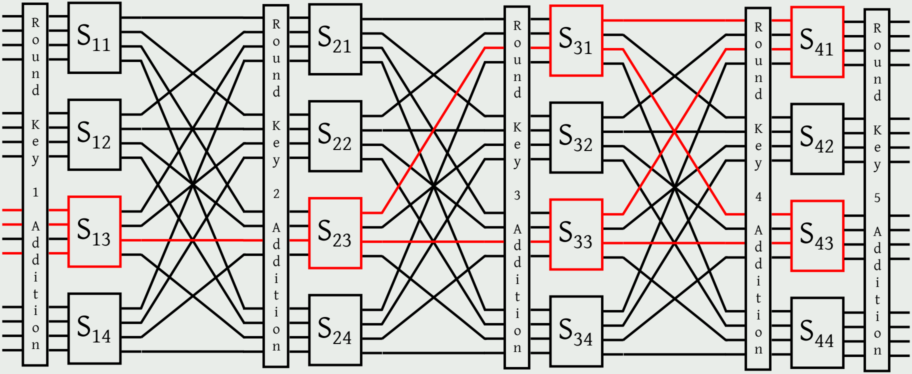
- Adding the fourth round key, and packing all the key bits in $\Sigma_k$ we get

$$r_4 : U_{4,6} \oplus U_{4,8} \oplus U_{4,14} \oplus U_{4,16} \oplus P_5 \oplus P_7 \oplus P_8 \oplus \Sigma_k = 0$$

  holding with a bias of $-\frac{1}{32}$, where $U_{4,i}$ are in input to the last s-box layer, and $P_j$ are the ptx bits.
- We now consider the sum of the involved key bits as a single value $\Sigma_k$: this can only assume 0 or 1 as its value
- **As the key is fixed, all the key bits have (at most) the net effect of flipping the sign of the relation bias which is thus $\pm\frac{1}{32}$**
- There we go :) We have a linear approximation, not dependent on the key bits, from the plaintext to the input of the fourth SBox layer!

$$U_{4,6} \oplus U_{4,8} \oplus U_{4,14} \oplus U_{4,16} \oplus P_5 \oplus P_7 \oplus P_8 = 0$$

## Complete linear approximation path



## Extracting the key bits for the last round

Consider a linear path along the first $r-1$ rounds of the cipher (holding with bias $\epsilon$)which involves some ptx bits as well as some bits coming from the penultimate key-addition layer

- Call *partial subkeys* the key bits of the last round-key influenced by the output bits of our approximation
- Collect $N \geq \frac{1}{\epsilon^2}$ ptx/ctx pairs [1]

1. Assume the partial subkey made of $L$ bits. For each partial subkey value, $sk_i \in \{0, 1, \ldots, 2^{L-1}\}$, count how many ptx/ctx pairs make the relation $r$ hold, say $N_{sk_i}$ this number (Use the ctxs to feed the inverse of the S-Boxes activated on the last layer and compute $U_{4,j}$)
2. Compute the biases $\frac{N_{sk_i}}{N} - \frac{1}{2}$ for each $sk_i$ value
3. Select the partial subkey value with the highest bias

Lather-rinse-repeat with other linear paths to get other partial subkeys

## Summary

1. Analyze the cipher Sboxes, computing the linear bias table
2. Choose viable relations for an S-Box (i.e., the ones with highest bias which activates as few sboxes as possible)
3. Propagate the biases up to the input of the last s-box layer employing the Pile-up lemma
4. Collect $\geq \frac{1}{\epsilon^2}$ ptx/ctx pairs
5. Check for all the possible partial subkeys involved in the linear approximation which one best fits
6. Repeat for other *partial subkeys* until the subkey of the last round is completely derived. The entire analysis can be trivially repeated to compute also the subkeys in other rounds, re-using the same ptx/ctx pairs as before.

# Further readings

## Practice and real world examples

- A `C` implementation of the whole attack and a computer for the linear biases is available on the website
- A real world cipher known to be terribly vulnerable to linear cryptanalysis is FEAL [2]:
  - it's only 4 rounds, so you can try and crack it
- For the more determined: you can read [1] to explore the full DES linear cryptanalysis

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